Institut Fourier — Université de Grenoble I

Actes du séminaire de

Théorie spectrale et géométrie

Yves Colin de Verdière

Elastic wave equation

Volume 25 (2006-2007), p. 55-69.

<http://tsg.cedram.org/item?id=TSG_2006-2007__25__55_0>

Séminaire de théorie spectrale et géométrie
GRENOBLE
Volume 25 (2006-2007) 55-69

ELASTIC WAVE EQUATION

Yves Colin de Verdière

ABSTRACT. — The goal of this talk is to describe the Lamé operator which drives the propagation of linear elastic waves. The main motivation for me is the work I have done in collaboration with Michel Campillo's group from LGIT (Grenoble) on passive imaging in seismology. From this work, several mathematical problems emerged: equipartition of energy between $S-$ and $P-$waves, high frequency description of surface waves in a stratified medium and related inverse spectral problems.

We discuss the following topics:

- What is the definition of the operator and the natural (free) boundary conditions?
- The polarizations of waves ($S-$waves and $P-$waves) and its relation to Hodge decomposition
- The Weyl law and equipartition of energy between $S-$waves and $P-$waves. We formulate here questions in the spirit of Schnirelman's Theorem about limits of Wigner measures of eigenmodes and of Schubert's Theorem about the large time equipartition of an evolved Lagrangian state.
- Rayleigh waves for the half-space: we compute in a rather explicit way the spectral decomposition following the work of Ph. Sécher. Of particular interest are the scattering matrix and the density of states.

1. Linear elastic waves

Let (X, g) be a 3-dimensional smooth compact Riemannian manifold *with boundary.* We want to define the *Lamé operator* L which is a linear self-adjoint elliptic differential operator of order 2 on the Hilbert space $L^2(X, TX)$ of L^2 vector fields u on X with respect to the density $\rho(x)|dx|$, i.e.:

$$\|u\|^2 = \int_X |u(x)|^2 \rho(x)|dx|$$

where the pointwise norm $|u(x)|$ is computed w.r. to the Riemannian metric. The vector field u corresponds to the small deplacement $x \to x + \varepsilon u(x)$ of the medium. The Dirichlet form (elastic energy induced by the small deplacement) is given as follows:

$$Q(u) = \int_X q_x(\delta u(x))|dx|$$

where $\delta u := \frac{1}{2}\mathcal{L}_u g$, the Lie derivative of g w.r. to u, is the *deformation tensor*. For example, if $g = dx_1^2 + dx_2^2 + dx_3^2$, $(\delta u)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. The quadratic form q_x is positive definite on the vector space $S^2(T_x^\star X)$. It means that q_x is a quadratic form on a vector space of dimension 6:

$$q_x(\delta u) = \sum_{i,j,k,l} c_{ij,kl}(\delta u)_{ij}(\delta u)_{kl} \ ,$$

which involves a priori 21 independent coefficients $c_{ij,kl},\ i,j,k,l = 1,2,3,$ with

- $c_{ij,kl} = c_{ji,kl}$
- $c_{ij,kl} = c_{ij,lk}$
- $c_{ij,kl} = c_{kl,ij}$.

Usually people do assume that q_x is isotropic, meaning that it is invariant by the natural action of $O(3)$ on T_xX. From the general results of invariant theory, it follows that $q_x(\delta u) = \lambda(x)\mathrm{Trace}(\delta u)^2 + 2\mu(x)\mathrm{Trace}(\delta u^2)$, i.e.

$$q_x(\delta u) = \lambda(x)\left(\sum_{i=1}^3 \frac{\partial u_i}{\partial x_i}\right)^2 + \frac{1}{2}\mu(x)\sum_{i,j=1}^3\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2,$$

the 21 coefficients $c_{ij,kl}$ reducing to 2. In order that q_x is positive definite we assume:

$$\mu > 0,\ \ \lambda > -\frac{2}{3}\mu \ . \tag{1.1}$$

The functions λ and μ are the so called *Lamé's coefficients.*

Lemma 1.1. — *The form $\mathcal{Q}$ defined on smooth vector fields on X is closable*[(1)].

Proof. — It is enough to proof that if $u_n \to 0$ in L^2 and δu_n converges to w in L^2, then $w = 0$: this is clear because $u_n \to 0$ in the sense of distributions and then δu_n too. □

Remark 1.2. — The domain of the closure of $\mathcal{Q}$ is the Sobolev space H^1 because $u \to \delta u$ is elliptic of order 1.

(1) It means that the completion of smooth vector fields w.r. to the norm $\|u\|_1^2 = \|u\|_{L^2} + \mathcal{Q}(u)$ is a subspace of L^2

It is known that if $\mathcal{Q}$ is a closable quadratic form on an Hilbert space, there is a canonical way to build from it a self-adjoint operator called its Friedrichs extension (see [10]).

Definition 1.3. — *The* Lamé operator L *is the linear self-adjoint operator on the Hilbert space of* L^2 *vector fields which is the* Friedrichs extension *of* Q .

We will see that L is elliptic et describe its domain (including the boundary conditions) in a precise way.

In order to simplify the calculations, we will assume that

- X is a smooth compact domain of $\mathbb{R}^3$ with the euclidian metric $dx_1^2 + dx_2^2 + dx_3^2$
- $\rho(x) \equiv 1$
- λ and μ are constants satisfying the inequalities (1.1).

2. Computing the Lamé operator L

We want to write in a rather explicit way the operator L and the "Neumann" (or free) boundary conditions for L.

We start with $Q(u) = \int_X q(du_1, du_2, du_3)|dx|$ where $u = u_1\partial_1 + u_2\partial_2 + u_3\partial_3$. Let us introduce the vector fields f_i, $i = 1, 2, 3$ defined by $f_i = \frac{1}{2}\partial q/\partial(du_i)$. The f_i's are vector fields because du_i is a $1-$form by duality (Legendre transform). The field f_i is the force created by the reaction of the elastic body to the deformation $u_i\partial_i$. We have(2)

$$Q(u,v) = \sum_i \int_X dv_i(f_i)|dx|,$$

$$Q(u,v) = \sum_i \int_X dv_i \wedge (\iota(f_i)dx)$$

and by integration by parts:

$$Q(u,v) = \sum_i \left(-\int_X v_i d(\iota(f_i)dx) + \int_{\partial X} v_i \iota(f_i)dx \right).$$

Hence

$$(Lu)_i dx = -d(\iota(f_i)dx) = -\mathcal{L}_{f_i} dx$$

(2) If α is a differential p-form and V a vector field, the inner product $\iota(V)\alpha$ is the $(p-1)$-form defined by putting V inside as the first entry

and the boundary conditions are given by:

$$\iota(f_i)dx = 0$$

as a differential form on the boundary ∂X; this boundary condition can be interpreted as saying that f_i is tangent to ∂X which is quite natural from the point of view of physics.

The symmetric tensor (f_1, f_2, f_3) is called the *stress tensor* and denoted $\sigma = (\sigma_{ij})$. Using an orthonormal frame, we get the expression:

$$\sigma_{ij} := \lambda \mathrm{div}(u)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Finally, we get:

$$L = -(\lambda + \mu)\mathrm{grad}\ \mathrm{div}\ u - \mu\Delta u$$

with the boundary conditions: $\sigma\nu = 0$ where ν is the normal to the boundary.

The principal symbol $l(x, \xi)$ of L can then be computed:

$$l(x, \xi) = (\lambda + \mu)(\xi_i \xi_j) + \mu\|\xi\|^2 \mathrm{Id}.$$

We see that L is *elliptic.*

We are interested in the wave equation

$$u_{tt} + Lu = 0$$

and the related spectral theory; there exists an orthonormal basis of the Hilbert space of $L^2(X, \mathbb{R}^3)$ which consists of eigenmodes $Lu_j = \omega_j^2 u_j$ with the usual convention

$$0 = \omega_1 = \cdots = \omega_6 < \omega_7 \leqslant \cdots \leqslant \omega_j \leqslant \cdots$$

where each eigenvalue is repeated according to its own multiplicity. Let us remark that $\omega_1 = 0$ is of multiplicity 6, the eigenspace being the space of infinitesimal isometries of $\mathbb{R}^3$ for which $\delta u = 0$ by definition.

3. S-waves and P-waves

For $(x, \xi) \in T^\star X$, $l(x, \xi)$ is a positive definite symmetric endomorphism of $T_x X$ which admits 2 eigenvalues: $\lambda_P(x, \xi) = (\lambda + 2\mu)\|\xi\|^2$ and $\lambda_S(x, \xi) = \mu\|\xi\|^2$. The corresponding eigenspaces are

$$E_P(x, \xi) = \mathbb{R}v(x, \xi),\ E_S(x, \xi) = v(x, \xi)^\perp$$

where $v(x, \xi) \in T_x X$ is the Legendre transform of (x, ξ). We will denote by π_P(resp. π_S) the orthogonal projectors of $\mathbb{R}^3$ onto E_P (resp. E_S). Let us remark that we have hence 2 dynamics on $T^\star X$ associated to the Hamiltonians λ_P and λ_S. The P(resp. S)-waves propagates following the Hamiltonian λ_P (resp. λ_S) and polarizations E_P (resp. E_S).

The speed of the P(resp. S)-waves is $c_P = \sqrt{\lambda + 2\mu}$ (resp. $c_S = \sqrt{\mu}$).

4. Link with Hodge Laplacians

4.1. Helmholtz decomposition

They are several ways to decompose a vector field into a sum of a gradient and a divergence free vector fields. This problem is fully described in the book [14].

Identifying vector fields and 1-forms, we have:

THEOREM 4.1. — *Every smooth vector field u in X can be uniquely written as*

$$u = u_P + u_S \ ,$$

with

- $u_P = \mathrm{grad}\ f$ *and f smooth*
- $\mathrm{div}\ u_S = 0$ *and u_S is tangent to ∂X*

Moreover, both parts are orthogonal. The associated L^2 projectors Π_P and Π_S are pseudo-differential operators of symbols $\pi_P(x,\xi)$ =the orthogonal projector onto $\mathbb{R}\xi$ and $\pi_S(x,\xi) = \mathrm{Id} - \pi_P$.

4.2. Lamé operator and Hodge theory

Using the Riemannian metric, we can identify vector fields with 1-forms. On the 1-forms, we have already the “natural” differential operators $\Delta_+ = dd^\star$ and $\Delta_- = d^\star d$. The Hodge laplacian is usually defined as $\Delta = \Delta_+ + \Delta_-$. We want to compute L in terms of Δ_+ and Δ_-. The main result is:

THEOREM 4.2. — *With the natural identification of vector fields and $1-$forms, the action of L on $C_o^\infty(X,\mathbb{R}^3)$ is given by:*

$$L = (\lambda + 2\mu)\Delta_+ + \mu\Delta_- \ .$$

In other words, $L(u_P + u_S) = (\lambda + 2\mu)\Delta u_P + \mu \Delta u_S$. The boundary conditions for L are not those given in Hodge theory.

The previous identity comes from $\Delta_+ \equiv -\mathrm{grad}\ \mathrm{div}$ and $\Delta_- \equiv \mathrm{rot}\ \mathrm{rot}$. Let us note that the eigenmodes have in general a non trivial decomposition $u_j = u_{j,P} + u_{j,S}$.

5. The spectral resolution of L in $\mathbb{R}^3$

Let us write the spectral measure of $\sqrt{L}$ with constant coefficients in $\mathbb{R}^3$:

$$\delta(\mathbf{x}=\mathbf{y}) = \int_0^\infty e(\mathbf{x},\mathbf{y},\omega)d\omega \ ,$$

and

$$[L](\mathbf{x},\mathbf{y}) = \int_0^\infty \omega^2 e(\mathbf{x},\mathbf{y},\omega)d\omega \ .$$

We insert $\mathrm{Id} = \pi_S(\mathbf{k}) + \pi_P(\mathbf{k})$ in the Fourier inversion formula:

$$\delta(\mathbf{x}=\mathbf{y}) = (2\pi)^{-3}\left(\int_{\mathbb{R}^3} e^{i\langle \mathbf{k}|\mathbf{x}-\mathbf{y}\rangle}\pi_P(\mathbf{k})d\mathbf{k} + \int_{\mathbb{R}^3} e^{i\langle \mathbf{k}|\mathbf{x}-\mathbf{y}\rangle}\pi_S(\mathbf{k})d\mathbf{k}\right).$$

Using polar coordinates $\mathbf{k} = c_P^{-1}\omega\vec{u}$ (with $|\vec{u}|=1$) and $\mathbf{k} = c_S^{-1}\omega\vec{u}$, we get:

$$e(\mathbf{x},\mathbf{y},\omega) =$$
$$\frac{\omega^2}{(2\pi)^3}\left(c_P^{-3}\int_{c_P k=\omega} e^{i\langle \mathbf{k}|\mathbf{x}-\mathbf{y}\rangle}\pi_P(\mathbf{k})d\theta + c_S^{-3}\int_{c_S k=\omega} e^{i\langle \mathbf{k}|\mathbf{x}-\mathbf{y}\rangle}\pi_S(\mathbf{k})d\theta\right),$$

where $d\theta$ is the uniform measure of total mass 4π on the unit sphere and $k := |\mathbf{k}|$.

We get hence the density of states $d\sigma(x,\omega) := \mathrm{trace}(e(x,x,\omega))d\omega$:

$$d\sigma(x,\omega) = \frac{1}{2\pi^2}\left(\frac{1}{c_P^3} + \frac{2}{c_S^3}\right)\omega^2 d\omega.$$

6. Explicit spectral decomposition in the case of the half-space

We will now consider the case of the half-space $X := \{(x,y,z)|z \leqslant 0\}$. We want to compute quite explicitely the spectral decomposition of the Lamé operator in X with constant Lamé coefficients. Using the invariance by translation, we can decompose L as an integral of operators $L_{\mathbf{k}}$, $\mathbf{k}\in\mathbb{R}^2$ which we need to study explicitely:

$$L = \int_{\mathbb{R}^2} \mathcal{F}^{-1} L_{\mathbf{k}} \mathcal{F} d\mathbf{k}. \tag{6.1}$$

As a consequence we get an explicit expression of the unitary reflection matrix $R(\mathbf{k})$.

Let us first describe roughly the situation: the operators $L_{\mathbf{k}}$ act on $L^2(\mathbb{R}^-,\mathbb{R}^3)$. The operator $L_{\mathbf{k}}$ splits into a trivial part $L'_{\mathbf{k}}$ corresponding to pure $S-$waves reflecting on the boundary with a polarization tangent to the boundary and a non trivial part $L''_{\mathbf{k}}$ with modes conversions. $L'_{\mathbf{k}}$ is equivalent to the scalar Laplace operator on the half space with Neumann boundary conditions.

The spectrum of $\sqrt{L''_k}$ splits into 3 parts:

- An eigenvalue of multiplicity 1, $c_R k$ with $0 < c_R < c_S$. The corresponding waves are called the *Rayleigh* waves
- A continuous spectrum of multiplicity 1, the intervall $I = [c_S k, c_P k]$ corresponding to pure reflection of $S-$waves with a reflection coefficient $r_{SS}(k)$ of modulus 1
- A continuous spectrum of multiplicity 2, the intervall $J = [c_P k, +\infty[$ corresponding to incident $S-$ or $P-$waves which are reflected as a linear combination of both kind of waves. We have then a reflection matrix which is a 2×2 unitary matrix.

The reflection matrix is also calculated in the paper [11].

6.1. Separation of variables: the Sturm-Liouville operator L_k.

We can reduce to the case where $\mathbf{k} = \begin{pmatrix} 0 \\ k \end{pmatrix}$. For $k \in \mathbb{R}$, let us compute

$$L_k u(z) := e^{-iky} L(e^{iky} u(z))$$

with $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

A short calculus gives:

$$L_k = \begin{pmatrix} \mu(k^2 + D^2) & 0 & 0 \\ 0 & (\lambda + 2\mu)k^2 + \mu D^2 & (\lambda + \mu)kD \\ 0 & (\lambda + \mu)kD & \mu k^2 + (\lambda + 2\mu)D^2 \end{pmatrix},$$

with $D = -i\frac{d}{dz}$.

The boundary conditions at $z = 0$ are:

$$\begin{cases} u'_1 & = 0 \\ u'_2 + iku_3 & = 0 \\ ik\lambda u_2 + (\lambda + 2\mu)u'_3 & = 0 \end{cases} .$$

We see that L_k splits as $L'_k + L''_k$ while splitting

$$u = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \\ u_3 \end{pmatrix} .$$

The spectral resolution of $\sqrt{L'_k}$ is easily related to the cosinus transform

$$f(z) = \frac{2}{\pi}\int_{-\infty}^{0} dz' \int_{0}^{+\infty} d\zeta \cos\zeta z \cos\zeta z' f(z') \ .$$

By performing the change $\zeta := \sqrt{\frac{\omega^2}{\mu} - k^2}$, we get:

$$de'_k(z,z',\omega) = \frac{2}{\pi}\frac{\omega}{c_S^2\zeta}\cos\zeta z \cos\zeta z' d\omega.$$

We are reduced to study L''_k.

6.2. L''_k on the real line

We want to describe the spectral decomposition on the half-line from that on the line as a scattering problem. Let us first write explicitely the spectral decomposition of L_k on the line. As in section 5, we start from Fourier inversion formula, but now with one variable. Following the same path, we get the following spectral resolution for L_k:

$$\delta(z,z') = \frac{1}{2\pi}$$
$$\left(\int_{kc_P}^{\infty} \frac{\omega}{c_P^2\zeta_P} e^{i\zeta_P(z-z')}\pi_P(0,k,\zeta_S)d\omega + \int_{kc_S}^{\infty} \frac{\omega}{c_S^2\zeta_S} e^{i\zeta_S(z-z')}\pi_S(0,k,\zeta_S)d\omega\right).$$

This decomposition gives the normalisation of the generalized eigenfunctions:

$$u_P^{\pm} := (\omega\zeta_P)^{-\frac{1}{2}} e^{\pm i\zeta_P z}\begin{pmatrix} k \\ \pm\zeta_P\end{pmatrix}$$

and

$$u_S^{\pm} := (\omega\zeta_S)^{-\frac{1}{2}} e^{\pm i\zeta_S z}\begin{pmatrix} \pm\zeta_S \\ -k\end{pmatrix}.$$

6.3. The reflection matrix

We now compute the reflection matrix

$$R_k(\omega) = \begin{pmatrix} r_{PP} & r_{PS} \\ r_{SP} & r_{SS}\end{pmatrix}$$

for $\omega^2 \in J$ and $r_{SS}(\omega)$ for $\omega^2 \in I$ by looking at eigenmodes of L_k of the form: $u'_\omega := u_P^+ + r_{PP}u_P^- + r_{PS}u_S^-$ satisfying the boundary condition. And similarly $u''_\omega := u_S^+ + r_{SP}u_P^- + r_{SS}u_S^-$.

We get the following result: if $A := (2k^2 - \frac{\omega^2}{\mu})$ and $B = 2k\sqrt{\zeta_P \zeta_S}$,

$$R_k(\omega) = \frac{1}{A^2 + B^2} \begin{pmatrix} B^2 - A^2 & 2AB \\ -2AB & B^2 - A^2 \end{pmatrix} .$$

We check that $R_k(\omega)$ is unitary in J, $|r_{SS}(\omega)| = 1$ in I and $R_k(\omega)$ admit poles at the zeroes of $\delta = A^2 + B^2$.

6.4. The Rayleigh waves

Let us look at the zeroes of $\delta = A^2 + B^2$ which are poles of $R_k(\omega)$: putting $\Omega = \omega/k\sqrt{\mu}$ and $r = \mu/(\lambda + 2\mu)$, we get

$$4\sqrt{1 - \rho\Omega^2)(1 - \Omega^2)} = (2 - \Omega^2)^2 \ .$$

This equation admits the root 0 and the square roots of the zeroes of a polynomial of degree 3: $p(t) = t^3 - 8t^2 + 8(3 - 2\rho)t - 16(1 - \rho)$. This polynomial admits a real root $0 < t_0 < 1$ and 2 non real roots. Let us define $c_R := c_S\sqrt{t_0}$. For $\omega = c_R k$, the residu of u' or of u'' is a wave exponentially decaying as $z \to -\infty$, called the Rayleigh wave: $u_{\mathrm{Rayl}} = u_P^- + \alpha u_S^-$. We will denote by u_R the L^2 normalized Rayleigh wave. Let us note that the polarisation of the Rayleigh wave is contained in the plane normal to the boundary containing the incoming ray (normal to the incident wave). Moreover, because then ζ_S and ζ_P are purely imaginary, the polarization is complex, meaning that we have a real polarization which lies on some ellipse: at a fixed point of the boundary, the wave is moving on an ellipse with a frequency kc_R.

6.5. The spectral resolution of L_k''

We are now able to write the spectral resolution of $\sqrt{L_k}$:

$$de_k(\omega) = de_{k,R} + de_{k,SN} + de_{k,ST} + de_{k,P}$$

with

$$de_{k,R}(z, z') = \delta(\omega = c_R k) u_R(z) \otimes (u_R(z'))^\star$$

$$de_{k,SN}(z, z') = \frac{2}{\pi} \mathbf{1}_{[c_S k, \infty[}(\omega) \frac{\omega}{c_S^2 \zeta_S} \cos\zeta_S z \cos\zeta_S z') \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\omega$$

$$de_{k,ST}(z, z') = \frac{1}{2\pi} \mathbf{1}_{[c_S k, \infty[}(\omega) u''_\omega(z) \otimes (u''_\omega(z'))^\star d\omega$$

$$de_{k,P}(z, z') = \frac{1}{2\pi} \mathbf{1}_{[c_P k, \infty[}(\omega) u'_\omega(z) \otimes (u'_\omega(z'))^\star d\omega.$$

6.6. The spectral resolution of L in $X = \{z \leqslant 0\}$

We can obtain the spectral resolution of L by puting the spectral resolutions of the L_k's into the formula (6.1).

7. Weyl law and equipartition of energy

7.1. Weyl law

Let us consider the spectral decomposition (u_j, ω_j) with $Lu_j = \omega_j^2 u_j$, $j = 1, \cdots$, and u_j an orthonormal basis of $L^2(X, \mathbb{R}^3)$.

THEOREM 7.1. — **(The Weyl law)** *If* $N(\omega) := \#\{j | \omega_j \leqslant \omega\}$ *we have* $N(\omega) \sim N_P(\omega) + N_S(\omega)$ *with*

- $N_P(\omega) := \frac{1}{6\pi^2}\mathrm{vol}(X)\left(\dfrac{\omega}{c_P}\right)^3$
- $N_S(\omega) := \frac{\mathbf{2}}{6\pi^2}\mathrm{vol}(X)\left(\dfrac{\omega}{c_S}\right)^3$

The previous theorem says that the asymptotic behaviour of the eigenvalues is the same as if we had 3 decoupled scalar operators, one of symbol $c_P^2\|\xi\|^2$ and 2 of symbol $c_S^2\|\xi\|^2$ acting on functions.

We have a more precise result (usually called the local Weyl law):

THEOREM 7.2. — *For any homogeneous symbol* $a \in C^\infty(T^\star X, \mathrm{Sym}(\mathbb{R}^3))$ *compactly supported in* $x \in X$,

$$\sum_{\omega_j \leqslant \omega} \langle \mathrm{Op}(a) u_j | u_j \rangle \sim (2\pi)^{-3}$$
$$\left(\int_{c_P|\xi| \leqslant \omega} \mathrm{trace}_P(a) |dx d\xi| + \int_{c_S|\xi| \leqslant \omega} \mathrm{trace}_S(a) |dx d\xi| \right),$$

with $\mathrm{trace}_P(a) = \mathrm{trace}(\pi_P a)$ *and similarly for* $\mathrm{trace}_S(a)$.

In particular

Corollary 7.3. — *For any domain $D \subset \mathrm{Int}(X)$, we have*

$$\sum_{\omega_j \leqslant \omega} \int_D \|u_{j,P}\|^2 |dx| \sim \frac{1}{6\pi^2} |D| \left(\frac{\omega}{c_P}\right)^3,$$

$$\sum_{\omega_j \leqslant \omega} \int_D \|u_{j,S}\|^2 |dx| \sim \frac{2}{6\pi^2} |D| \left(\frac{\omega}{c_S}\right)^3,$$

$$\sum_{\omega_j \leqslant \omega} \int_D \langle u_{j,P} | u_{j,S} \rangle |dx| = O(\omega^2).$$

A very basic question is to understand how the dynamics produces a random wave in the previous setting. Let us give a

Definition 7.4. — *Let us take a solution of the wave equation $u(x,t)$ and a frequency Φ_ω cut-off around the frequency ω. We will say that $u(x,t)$ is* equipartited *if we have for large ω:*

$$\lim_{t \to \infty} \frac{\int_D \|\Phi_\omega u(x,t)\|^2 |dx|}{\int_X \|\Phi_\omega u(x,t)\|^2 |dx|} = \frac{|D|}{|X|},$$

$$\lim_{t \to \infty} \frac{\int_D \|\Phi_\omega u_S(x,t)\|^2 |dx|}{\int_D \|\Phi_\omega u_P(x,t)\|^2 |dx|} = \frac{2c_P^3}{c_S^3}.$$

Remark 7.5. — From the experimental point of view, only the second limit can be measured.

There are in fact 2 cases:

(1) The case $u = u_j(x) e^{i\omega_j t}$. Do the large frequencies individual eigenmodes satisfy an equipartition property?
(2) The case where the Cauchy datas of u are localized at some point x_0. This case is much more interesting from the physical point of view. It corresponds to an earthquake with a source at x_0.

In the case of the (scalar) Laplace operator, both cases of the question have satisfactory answers:

- The first one is the celebrated Schnirelman Theorem (1974) which says that the answer is yes if the geodesic (billiard) flow is *ergodic* [12, 16, 3, 5, 17]
- The second one is a recent result by Roman Schubert [13] which says that the answer is yes if the geodesic (billiard) flow is *Anosov* and if t is of the order of the so-called Heisenberg time. It uses the fact that Anosov systems are mixing.

7.2. A statistical interpretation of Weyl laws: the microcanonical ensemble

On any Hilbert space $H = (\mathcal{H}, \langle .|.\rangle)$, there is a canonical random field w_H, the white noise of H, which satisfies:

$$\forall e, f \in \mathcal{H},\ \mathbb{E}(\langle w_H|e\rangle \overline{\langle w_H|f\rangle}) = \langle f|e\rangle \ .$$

If $\dim \mathcal{H} = \infty$, this field is not a vector of $\mathcal{H}$ but a "distribution": if $A : H \to K$ is an Hilbert-Schmidt operator, Aw_H is a random field on K.

Let us give some $E > 0$ and consider the Hilbert space $H_E \subset L^2(X, TX)$ which is generated by the eigenfields of frequency less than E. The associated random fields w_E can be splited as $w_E = w_{E,S} + w_{E,P}$.

8. The classical limit

Let us fix the frequency $\omega > 0$. Inside X, we have 2 decoupled dynamics: they are given by the Hamiltonians $h_P = c_P\|\xi\|$ (resp. $h_S = c_S\|\xi\|$) on the energy surfaces $\Sigma_P = \{h_P = \omega\}$ (resp. $\Sigma_S = \{h_S = \omega\}$). More precisely, we will introduce an extended phase space $Z := Z_S \cup Z_P$ with a measure dm where

- Z_S is the total space of the projective fiber bundle $P(E_S) \to \Sigma_S$ with the dynamics given by $\Phi_t^S(z, < v >) = (\phi_t^S(z), < T_t(v) >)$ with ϕ_t^S the Hamiltonian flow of h_S and T_t the parallel transport in the fiber bundle E_S (the "Berry" phase). Z_S is equiped with the measure $dm_S = dL_S \otimes dh$ where dL_S is the microcanonical Liouville measure and dh is the uniform measure on $P(E_S)$ with total mass 2.
- Similarly $Z_P = \Sigma_P$ and dm_P is the microcanonical Liouville measure and the Hamiltonian dynamics φ_t^P associated to h_P.

Of course the previous dynamics is not yet defined when hitting the boundary. At that point, we will define it as a Markov process whose probability transitions are given by the squares π_P, π_S of the entries of the reflection matrix. Let us remark that the component of an $S-$wave tangent to the boundary (and normal to the ray) is reflected without adding a $P-$component. On the other hand the incoming and outgoing points $(y, \xi^\pm)$, $y \in \partial X$ are related by $\xi^+_{|T_yX} = \xi^-_{|T_yX}$.

Let us check that the measure dm is invariant by the previous stochastic process. Let us assume that $X_0(\omega)$ follows the probabiliy dm:

$\mathbb{P}(\{X_0 \in U\}) = m(U)$ for any $U \subset Z$. We need to check that, for all $t > 0$,

$$\mathbb{P}(\{X_t \in U\}) = m(\{X_0 \in U\}) \ .$$

Let us do that for small t and U: let us consider for example $U \subset Z_P$ and t so that the only way that $X_t \in U$ is one reflection at some point $y \in \partial X$. Either $X_0 \in U_-^S$ or $X_0 \in U_-^P$. In the first case $X_0 = (z_0, \theta)$ and we can fix that $\theta = 0$ corresponds to a polarisation in the plane generated by the ray and the normal to the boundary. We get

$$\mathbb{P}(\{X_t \in U\}) = \int_{U_-^P} \pi_P dm_P + \frac{2}{\pi} \int_{U_-^S} \pi_S \cos^2 \theta dL_S d\theta \ .$$

We get, using

$$\frac{2}{\pi} \int_0^\pi \cos^2 \theta \ d\theta = 1,$$

$$\mathbb{P}(\{X_t \in U\}) = \int_{U_-^P} \pi_P dm_P + \int_{U_-^S} \pi_S dm_S$$

and using the fact that the microlocal Liouville measures are conserved by the reflection process and that $\pi_P + \pi_S = 1$, we get the result. One can check a similar result if $U \subset Z_S$.

In order to use the usual definitions of dynamical systems, we introduce the set $\mathcal{T}$ of trajectories $z(t)$, $t \in \mathbb{R}$ with the measure dP given as usual on cylindrical sets. The measure dP is invariant by the translation along trajectories. So we can say that the classical limit is ergodic or mixing.

8.1. Conjectures

- In the spirit of Schnirelman's Theorem, we can ask the following:

  Question 8.1. — *If the classical random walk is ergodic, do we have equipartition of energies for a density one subsequence of eigenmodes u_{j_l}?*

  Let us remark that the previous property is not valid if the manifold has no boundary. This conjecture is probably not true. A similar conjecture for another model with a probabilistic classical dynamics is not valid: the case of the Laplace operator on a generic star graph for which Schnirelman Theorem does not hold (see [7] for explicit formulae for eigenfunctions). At least the usual proof uses Egorov's Theorem which is not true in this context!

- In the spirit of Schubert Theorem, I put the following

Conjecture 8.2. — *Let us assume that the classical limit is mixing and has some kind of hyperbolicity. If the Cauchy data of the Lamé wave equation are localized, there is a dynamical equipartition (large time): $u(x,t)$ satisfies equipartition for t large but smaller than Ehrenfest times (to be defined).*

9. Appendix A: spectral resolutions, density of states and microcanonical ensembles

Let L be a positive self-adjoint operator on $L^2(X,\mathbb{R}^N)$. The spectral resolution $de(x,y,\omega)$ of $\sqrt{L}$ is a measure with values in $\mathrm{Sym}(\mathbb{R}^N)$ which satisfies:

$$\delta(x-y)=\int_{\mathbb{R}} de(x,y,\omega)$$

and, for any function Φ of L:

$$[\Phi(L)](x,y)=\int_{\mathbb{R}}\Phi(\omega^2)de(x,y,\omega).$$

The density of states is a measure on $X\times\mathbb{R}$ defined by $d\sigma(x,\omega)=\mathrm{trace}(de(x,x,\omega))$. In particular, we have, in case of a discrete spectrum:

$$\#\{\omega_j\leqslant\omega\}=\int_X dx\int_0^{\omega} d\sigma(x,\omega').$$

The *microcanonical ensemble* associated to an intervall $I=[\omega_-,\omega_+]$ is the white noise on the Hilbert space $\Pi_I(L^2(X,\mathbb{R}^N)$. We have then

$$\mathbb{E}(\int_D |u(x)|^2dx)=\int_D dx\int_{\omega\in I} d\sigma(x,\omega)\ .$$

In other words, the expected energy of a mode whose frequency lies in $[\omega_-,\omega_+]$ is given in terms of the density of states.

More generally, we have

$$\mathbb{E}_I(\langle Au|u\rangle)=\mathrm{Trace}A\Pi_I\ .$$

This can be evaluated in the large frequency regime if A is a ΨDO as the trace of the pseudo-differential operator $A\Pi_I$ of symbol $a\circ\chi_I(\sqrt{L})$.

BIBLIOGRAPHY

[1] J. Bolte & R. Glaser. Quantum ergodicity for Pauli Hamiltonians with spin 1/2. *Nonlinearity,* 13:1987–2003 (2000).

[2] J. Chazarain. Construction de la paramétrix du problème mixte hyperbolique pour l'équation des ondes. *C. R. Acad. Sci., Paris,* Sér. A 276:1213–1215 (1973).

[3] Yves Colin de Verdière. Ergodicité et fonctions propres du laplacien. *Commun. Math. Phys.,* 102:497–502 (1985).

[4] F. Faure & S. Nonnenmacher. On the maximal scarring for quantum cat map eigenstates. *Commun. Math. Phys.,* 245:201–214 (2004).

[5] P. Gérard & E. Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.,* 71:559–607 (1993).

[6] R. Hennino, N. Trégourès, N. M. Shapiro, L. Margerin, M. Campillo, B. A. van Tiggelen & R. L. Weaver. Observation of Equipartition of Seismic Waves. *Phys. Rev. Lett.,* 86:3447–3450 (2001).

[7] J.P. Keating, J. Marklof & B. Winn. Value Distribution of the Eignefunctions and Spectral Determinants of Quantum Star Graphs. *Commun. Math. Phys.,* 241:421–452 (2003).

[8] L. Landau & E. Lifchitz. Théorie de l'élasticité. *Editions Mir,* Moscow, 1990.

[9] L. Margerin, in preparation .

[10] M. Reed & B. Simon. Methods of Modern Mathematical Physics, vol 1. *Academic Press,* (1972).

[11] Ph. Sécher. Etude spectrale du système différentiel 2×2 associé à un problème d'élasticité linéaire. *Ann. Fac. Sc. Toulouse,* 7:699–726 (1998).

[12] A. Schnirelman, Ergodic properties of eigenfunctions. *Usp. Math. Nauk.,* 29:181–182 (1974).

[13] R. Schubert. Semi-classical Behaviour of Expectation Values in Time Evolved Lagrangian States for Large Times. *Commun. Math. Phys.,* 256:239–254 (2005).

[14] Günter Schwarz. Hodge Decomposition–A Method for Solving Boundary Values Problems. *Springer LN,* 1607 (1995).

[15] M. Taylor. Rayleigh waves in linear elasticity as a propagation of singularities phenomenon. *Partial differential equations and geometry (Proc. Conf., Park City, Utah, 1977), pp. 273–291, Lecture Notes in Pure and Appl. Math., 48, Dekker,* New York, 1979.

[16] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.,* 55:919–941 (1987).

[17] S. Zelditch & M. Zworski. Ergodicity of eigenfunctions for ergodic billiards. *Commun. Math. Phys.,* 175:673–682 (1996).

Yves Colin de Verdière
Université Grenoble 1
Institut Fourier — UMR CNRS-UJF 5582
BP 74
38402-Saint Martin d'Hères cedex (France)
yves.colin-de-verdiere@ujf-grenoble.fr