ELASTIC WAVE EQUATION

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Abstract. — The goal of this talk is to describe the Lamé operator which drives the propagation of linear elastic waves. The main motivation for me is the work I have done in collaboration with Michel Campillo’s group from LGIT (Grenoble) on passive imaging in seismology. From this work, several mathematical problems emerged: equipartition of energy between $S$– and $P$–waves, high frequency description of surface waves in a stratified medium and related inverse spectral problems.

We discuss the following topics:

• What is the definition of the operator and the natural (free) boundary conditions?
• The polarizations of waves ($S$–waves and $P$–waves) and its relation to Hodge decomposition
• The Weyl law and equipartition of energy between $S$–waves and $P$–waves. We formulate here questions in the spirit of Schnirelman’s Theorem about limits of Wigner measures of eigenmodes and of Schubert’s Theorem about the large time equipartition of an evolved Lagrangian state.
• Rayleigh waves for the half-space: we compute in a rather explicit way the spectral decomposition following the work of Ph. Sécher. Of particular interest are the scattering matrix and the density of states.

1. Linear elastic waves

Let $(X,g)$ be a 3-dimensional smooth compact Riemannian manifold with boundary. We want to define the Lamé operator $L$ which is a linear self-adjoint elliptic differential operator of order 2 on the Hilbert space $L^2(X,TX)$ of $L^2$ vector fields $u$ on $X$ with respect to the density $\rho(x)|dx|$, i.e.:

$$\|u\|^2 = \int_X |u(x)|^2 \rho(x)|dx|$$

where the pointwise norm $|u(x)|$ is computed w.r. to the Riemannian metric. The vector field $u$ corresponds to the small displacement $x \rightarrow x + \varepsilon u(x)$ of the medium. The Dirichlet form (elastic energy induced by the small
deplacement) is given as follows:

\[ Q(u) = \int_X q_x(\delta u(x))|dx| \]

where \( \delta u := \frac{1}{2} \mathcal{L}_u g \), the Lie derivative of \( g \) w.r. to \( u \), is the deformation tensor. For example, if \( g = dx_1^2 + dx_2^2 + dx_3^2 \), \( (\delta u)_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \). The quadratic form \( q_x \) is positive definite on the vector space \( S^2(T^*_x X) \). It means that \( q_x \) is a quadratic form on a vector space of dimension 6:

\[ q_x(\delta u) = \sum_{i,j,k,l} c_{ij,kl} (\delta u)_{ij} (\delta u)_{kl} \]

which involves a priori 21 independent coefficients \( c_{ij,kl} \), \( i, j, k, l = 1, 2, 3 \), with

- \( c_{ij,kl} = c_{ji,kl} \)
- \( c_{ij,kl} = c_{ij,lk} \)
- \( c_{ij,kl} = c_{kl,ij} \).

Usually people do assume that \( q_x \) is isotropic, meaning that it is invariant by the natural action of \( O(3) \) on \( T^*_x X \). From the general results of invariant theory, it follows that \( q_x(\delta u) = \lambda(x)\text{Trace}(\delta u)^2 + 2\mu(x)\text{Trace}(\delta u^2) \), i.e.

\[ q_x(\delta u) = \lambda(x) \left( \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \right)^2 + \frac{1}{2} \mu(x) \sum_{i,j=1}^{3} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 , \]

the 21 coefficients \( c_{ij,kl} \) reducing to 2. In order that \( q_x \) is positive definite we assume:

\[ \mu > 0, \quad \lambda > -\frac{2}{3} \mu . \]

The functions \( \lambda \) and \( \mu \) are the so called Lamé’s coefficients.

**Lemmas 1.1.** — The form \( Q \) defined on smooth vector fields on \( X \) is closable\(^{(1)} \).

**Proof.** — It is enough to proof that if \( u_n \to 0 \) in \( L^2 \) and \( \delta u_n \) converges to \( w \) in \( L^2 \), then \( w = 0 \): this is clear because \( u_n \to 0 \) in the sense of distributions and then \( \delta u_n \) too. \( \square \)

**Remark 1.2.** — The domain of the closure of \( Q \) is the Sobolev space \( H^1 \) because \( u \to \delta u \) is elliptic of order 1.

\(^{(1)}\) It means that the completion of smooth vector fields w.r. to the norm \( \|u\|^2_1 = \|u\|_{L^2} + Q(u) \) is a subspace of \( L^2 \).
It is known that if $Q$ is a closable quadratic form on an Hilbert space, there is a canonical way to build from it a self-adjoint operator called its Friedrichs extension (see \cite{10}).

**Definition 1.3.** — The Lamé operator $L$ is the linear self-adjoint operator on the Hilbert space of $L^2$ vector fields which is the Friedrichs extension of $Q$.

We will see that $L$ is elliptic et describe its domain (including the boundary conditions) in a precise way.

In order to simplify the calculations, we will assume that

- $X$ is a smooth compact domain of $\mathbb{R}^3$ with the euclidian metric $dx_1^2 + dx_2^2 + dx_3^2$.
- $\rho(x) \equiv 1$.
- $\lambda$ and $\mu$ are constants satisfying the inequalities (1.1).

## 2. Computing the Lamé operator $L$

We want to write in a rather explicit way the operator $L$ and the “Neumann” (or free) boundary conditions for $L$. We start with $Q(u) = \int_X q(du_1, du_2, du_3)|dx|$ where $u = u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3$. Let us introduce the vector fields $f_i$, $i = 1, 2, 3$ defined by $f_i = \frac{1}{2} \partial q/\partial (du_i)$. The $f_i$'s are vector fields because $du_i$ is a 1-form by duality (Legendre transform). The field $f_i$ is the force created by the reaction of the elastic body to the deformation $u_i \partial_i$. We have\(^{(2)}\)

\[ Q(u, v) = \sum_i \int_X dv_i(f_i)|dx|, \]

\[ Q(u, v) = \sum_i \int_X dv_i \wedge (\iota(f_i)dx) \]

and by integration by parts:

\[ Q(u, v) = \sum_i \left( -\int_X v_i d(\iota(f_i))dx + \int_{\partial X} v_i \iota(f_i)dx \right). \]

Hence

\[ (Lu)_i dx = -d(\iota(f_i))dx = -L_f_i dx \]

\(^{(2)}\) If $\alpha$ is a differential $p$-form and $V$ a vector field, the inner product $\iota(V)\alpha$ is the $(p-1)$-form defined by putting $V$ inside as the first entry.
and the boundary conditions are given by:
\[ \nu(f_i)dx = 0 \]
as a differential form on the boundary \( \partial X \); this boundary condition can be interpreted as saying that \( f_i \) is tangent to \( \partial X \) which is quite natural from the point of view of physics.

The symmetric tensor \((f_1, f_2, f_3)\) is called the stress tensor and denoted \( \sigma = (\sigma_{ij}) \). Using an orthonormal frame, we get the expression:
\[ \sigma_{ij} := \lambda \text{div}(u) \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]

Finally, we get:
\[ L = -(\lambda + \mu) \text{grad} \text{ div} u - \mu \Delta u \]
with the boundary conditions: \( \sigma \nu = 0 \) where \( \nu \) is the normal to the boundary.

The principal symbol \( l(x, \xi) \) of \( L \) can then be computed:
\[ l(x, \xi) = (\lambda + \mu) (\xi_i \xi_j) + \mu \| \xi \|^2 \text{Id}. \]
We see that \( L \) is elliptic.

We are interested in the wave equation
\[ u_{tt} + Lu = 0 \]
and the related spectral theory; there exists an orthonormal basis of the Hilbert space of \( L^2(X, \mathbb{R}^3) \) which consists of eigenmodes \( Lu_j = \omega_j^2 u_j \) with the usual convention
\[ 0 = \omega_1 = \cdots = \omega_6 < \omega_7 \leq \cdots \leq \omega_j \leq \cdots \]
where each eigenvalue is repeated according to its own multiplicity. Let us remark that \( \omega_1 = 0 \) is of multiplicity 6, the eigenspace being the space of infinitesimal isometries of \( \mathbb{R}^3 \) for which \( \delta u = 0 \) by definition.

### 3. S-waves and P-waves

For \((x, \xi) \in T^*X \), \( l(x, \xi) \) is a positive definite symmetric endomorphism of \( T_x X \) which admits 2 eigenvalues: \( \lambda_P(x, \xi) = (\lambda + 2\mu) \| \xi \|^2 \) and \( \lambda_S(x, \xi) = \mu \| \xi \|^2 \). The corresponding eigenspaces are
\[ E_P(x, \xi) = \mathbb{R} v(x, \xi), \ E_S(x, \xi) = v(x, \xi)^\perp \]
where \( v(x, \xi) \in T_x X \) is the Legendre transform of \((x, \xi)\). We will denote by \( \pi_P \) (resp. \( \pi_S \)) the orthogonal projectors of \( \mathbb{R}^3 \) onto \( E_P \) (resp. \( E_S \)). Let us
remark that we have hence 2 dynamics on $T^*X$ associated to the Hamiltonians $\lambda_P$ and $\lambda_S$. The $P$(resp. $S$)-waves propagates following the Hamiltonian $\lambda_P$ (resp. $\lambda_S$) and polarizations $E_P$ (resp. $E_S$).

The speed of the $P$(resp. $S$)-waves is $c_P = \sqrt{\lambda + 2\mu}$ (resp. $c_S = \sqrt{\mu}$).

4. Link with Hodge Laplacians

4.1. Helmholtz decomposition

They are several ways to decompose a vector field into a sum of a gradient and a divergence free vector fields. This problem is fully described in the book [14].

Identifying vector fields and 1-forms, we have:

**THEOREM 4.1.** — Every smooth vector field $u$ in $X$ can be uniquely written as $$u = u_P + u_S,$$
with

- $u_P = \text{grad } f$ and $f$ smooth
- $\text{div } u_S = 0$ and $u_S$ is tangent to $\partial X$

Moreover, both parts are orthogonal. The associated $L^2$ projectors $\Pi_P$ and $\Pi_S$ are pseudo-differential operators of symbols $\pi_P(x, \xi) = \text{the orthogonal projector onto } \mathbb{R}\xi$ and $\pi_S(x, \xi) = \text{Id} - \pi_P$.

4.2. Lamé operator and Hodge theory

Using the Riemannian metric, we can identify vector fields with 1-forms. On the 1-forms, we have already the "natural" differential operators $\Delta_+ = dd^*$ and $\Delta_- = d^*d$. The Hodge laplacian is usually defined as $\Delta = \Delta_+ + \Delta_-$. We want to compute $L$ in terms of $\Delta_+$ and $\Delta_-$. The main result is:

**THEOREM 4.2.** — With the natural identification of vector fields and 1–forms, the action of $L$ on $C^\infty_0(X, \mathbb{R}^3)$ is given by:

$$L = (\lambda + 2\mu)\Delta_+ + \mu\Delta_-.$$ 

In other words, $L(u_P + u_S) = (\lambda + 2\mu)\Delta u_P + \mu\Delta u_S$. The boundary conditions for $L$ are not those given in Hodge theory.

The previous identity comes from $\Delta_+ \equiv -\text{grad div}$ and $\Delta_- \equiv \text{rot rot}$. Let us note that the eigenmodes have in general a non trivial decomposition $u_j = u_{j,P} + u_{j,S}$. 
5. The spectral resolution of $L$ in $\mathbb{R}^3$

Let us write the spectral measure of $\sqrt{L}$ with constant coefficients in $\mathbb{R}^3$:

$$\delta(x = y) = \int_0^\infty e(x, y, \omega)d\omega,$$

and

$$[L](x, y) = \int_0^\infty \omega^2 e(x, y, \omega)d\omega.$$ 

We insert $\text{Id} = \pi_S(k) + \pi_P(k)$ in the Fourier inversion formula:

$$\delta(x = y) = (2\pi)^{-3}\left(\int_{\mathbb{R}^3} e^{i(k|x-y|)}\pi_P(k)dk + \int_{\mathbb{R}^3} e^{i(k|x-y|)}\pi_S(k)dk\right).$$

Using polar coordinates $k = c_P^{-1}\omega \vec{u}$ (with $|\vec{u}| = 1$) and $k = c_S^{-1}\omega \vec{u}$, we get:

$$e(x, y, \omega) = \frac{\omega^2}{(2\pi)^3}\left(c_P^{-3}\int_{c_Pk=\omega} e^{i(k|x-y|)}\pi_P(k)d\theta + c_S^{-3}\int_{c_Sk=\omega} e^{i(k|x-y|)}\pi_S(k)d\theta\right),$$

where $d\theta$ is the uniform measure of total mass $4\pi$ on the unit sphere and $k := |k|$.

We get hence the density of states $d\sigma(x, \omega) := \text{trace}(e(x, x, \omega))d\omega$:

$$d\sigma(x, \omega) = \frac{1}{2\pi^2}\left(\frac{1}{c_P^3} + \frac{2}{c_S^3}\right)\omega^2d\omega.$$

6. Explicit spectral decomposition in the case of the half-space

We will now consider the case of the half-space $X := \{(x, y, z)|z \leq 0\}$. We want to compute quite explicitly the spectral decomposition of the Lamé operator in $X$ with constant Lamé coefficients. Using the invariance by translation, we can decompose $L$ as an integral of operators $L_k, k \in \mathbb{R}^2$ which we need to study explicitly:

$$(6.1) \quad L = \int_{\mathbb{R}^2} F^{-1}L_kFd\mathbf{k}.$$ 

As a consequence we get an explicit expression of the unitary reflection matrix $R(k)$.

Let us first describe roughly the situation: the operators $L_k$ act on $L^2(\mathbb{R}^-, \mathbb{R}^3)$. The operator $L_k$ splits into a trivial part $L'_k$ corresponding to pure $S$-waves reflecting on the boundary with a polarization tangent
to the boundary and a non trivial part $L''_k$ with modes conversions. $L'_k$ is equivalent to the scalar Laplace operator on the half space with Neumann boundary conditions.

The spectrum of $\sqrt{L''_k}$ splits into 3 parts:

- An eigenvalue of multiplicity 1, $c_R k$ with $0 < c_R < c_S$. The corresponding waves are called the Rayleigh waves
- A continuous spectrum of multiplicity 1, the intervall $I = [c_S k, c_P k]$ corresponding to pure reflection of $S$-waves with a reflection coefficient $r_{SS}(k)$ of modulus 1
- A continuous spectrum of multiplicity 2, the intervall $J = [c_P k, +\infty[$ corresponding to incident $S$- or $P$-waves which are reflected as a linear combination of both kind of waves. We have then a reflection matrix which is a $2 \times 2$ unitary matrix.

The reflection matrix is also calculated in the paper [11].


We can reduce to the case where $k = \begin{pmatrix} 0 \\ k \end{pmatrix}$. For $k \in \mathbb{R}$, let us compute

$$L_k u(z) := e^{-iky}L(e^{iky}u(z))$$

with $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

A short calculus gives:

$$L_k = \begin{pmatrix} 
\mu(k^2 + D^2) & 0 & 0 \\
0 & (\lambda + 2\mu)k^2 + \mu D^2 & (\lambda + \mu)kD \\
0 & (\lambda + \mu)kD & \mu k^2 + (\lambda + 2\mu)D^2
\end{pmatrix},$$

with $D = -i \frac{d}{dz}$.

The boundary conditions at $z = 0$ are:

$$\begin{cases}
 u_1' = 0 \\
u_2' + iku_3 = 0 \\
 ik\lambda u_2 + (\lambda + 2\mu)u_3' = 0
\end{cases}.$$

We see that $L_k$ splits as $L'_k + L''_k$ while splitting

$$u = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \\ u_3 \end{pmatrix}.$$
The spectral resolution of $\sqrt{L_k'}$ is easily related to the cosinus transform

$$f(z) = \frac{2}{\pi} \int_{-\infty}^{0} dz' \int_{0}^{+\infty} d\zeta \cos \zeta z \cos \zeta' f(z').$$

By performing the change $\zeta := \sqrt{\frac{\omega^2}{c^2} - k^2}$, we get:

$$de_k'(z, z', \omega) = \frac{2}{\pi} \frac{\omega}{c^2_S \zeta} \cos \zeta z \cos \zeta' z' d\omega.$$

We are reduced to study $L_k''$.

### 6.2. $L_k''$ on the real line

We want to describe the spectral decomposition on the half-line from that on the line as a scattering problem. Let us first write explicitly the spectral decomposition of $L_k$ on the line. As in section 5, we start from Fourier inversion formula, but now with one variable. Following the same path, we get the following spectral resolution for $L_k$:

$$\delta(z, z') = \frac{1}{2\pi} \left( \int_{-k}^{k} \frac{\omega}{c^2_P \zeta} e^{i\zeta_P (z-z')} \pi_P(0, k, \zeta_P) d\omega + \int_{-k}^{k} \frac{\omega}{c^2_S \zeta} e^{i\zeta_S (z-z')} \pi_S(0, k, \zeta_S) d\omega \right).$$

This decomposition gives the normalisation of the generalized eigenfunctions:

$$u_{\pm}^P := (\omega \zeta_P)^{-\frac{1}{2}} e^{\pm i \zeta_P z} \begin{pmatrix} k \\ \pm \zeta_P \end{pmatrix}$$

and

$$u_{\pm}^S := (\omega \zeta_S)^{-\frac{1}{2}} e^{\pm i \zeta_S z} \begin{pmatrix} \pm \zeta_S \\ -k \end{pmatrix}.$$

### 6.3. The reflection matrix

We now compute the reflection matrix

$$R_k(\omega) = \begin{pmatrix} r_{PP} & r_{PS} \\ r_{SP} & r_{SS} \end{pmatrix}$$

for $\omega^2 \in J$ and $r_{SS}(\omega)$ for $\omega^2 \in I$ by looking at eigenmodes of $L_k$ of the form: $u_\omega' := u_\omega^P + r_{PP} u_P^P + r_{PS} u_S^P$ satisfying the boundary condition. And similarly $u_\omega'' := u_\omega^S + r_{SP} u_P^S + r_{SS} u_S^S$. 

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We get the following result: if $A := (2k^2 - \omega^2/\mu)$ and $B = 2k\sqrt{\zeta_P\zeta_S}$, 

$$R_k(\omega) = \frac{1}{A^2 + B^2} \begin{pmatrix} B^2 - A^2 & 2AB \\ -2AB & B^2 - A^2 \end{pmatrix}. $$

We check that $R_k(\omega)$ is unitary in $J$, $|r_{SS}(\omega)| = 1$ in $I$ and $R_k(\omega)$ admit poles at the zeroes of $\delta = A^2 + B^2$.

### 6.4. The Rayleigh waves

Let us look at the zeroes of $\delta = A^2 + B^2$ which are poles of $R_k(\omega)$: putting $\Omega = \omega/k\sqrt{\mu}$ and $r = \mu/(\lambda + 2\mu)$, we get 

$$4\sqrt{1 - \rho\Omega^2}(1 - \Omega^2) = (2 - \Omega^2)^2.$$

This equation admits the root 0 and the square roots of the zeroes of a polynomial of degree 3: $p(t) = t^3 - 8t^2 + 8(3 - 2\rho)t - 16(1 - \rho)$. This polynomial admits a real root $0 < t_0 < 1$ and 2 non real roots. Let us define $c_R := c_S\sqrt{t_0}$. For $\omega = c_R k$, the residu of $u'$ or of $u''$ is a wave exponentially decaying as $z \to -\infty$, called the Rayleigh wave: $u_{Rayl} = u_R$ for $\omega = c_R k$. We will denote by $u_R$ the $L^2$ normalized Rayleigh wave. Let us note that the polarisation of the Rayleigh wave is contained in the plane normal to the boundary containing the incoming ray (normal to the incident wave). Moreover, because then $\zeta_S$ and $\zeta_P$ are purely imaginary, the polarization is complex, meaning that we have a real polarization which lies on some ellipse: at a fixed point of the boundary, the wave is moving on an ellipse with a frequency $kc_R$.

### 6.5. The spectral resolution of $L''_k$

We are now able to write the spectral resolution of $\sqrt{L_k}$: 

$$de_k(\omega) = de_{k,R} + de_{k,SN} + de_{k,ST} + de_{k,P}$$

with 

$$de_{k,R}(z, z') = \delta(\omega = c_R k)u_R(z) \otimes (u_R(z'))^*$$

$$de_{k,SN}(z, z') = \frac{2}{\pi} 1_{[c_S, \infty]}(\omega) \frac{\omega}{c_S^2 \zeta_S} \cos \zeta_S z \cos \zeta_S z' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\omega$$

$$de_{k,ST}(z, z') = \frac{1}{2\pi} 1_{[c_S, \infty]}(\omega) u''_\omega(z) \otimes (u''_\omega(z'))^* d\omega$$

$$de_{k,P}(z, z') = \frac{1}{2\pi} 1_{[c_P, \infty]}(\omega) u'_\omega(z) \otimes (u'_\omega(z'))^* d\omega.$$
6.6. The spectral resolution of $L$ in $X = \{ z \leq 0 \}$

We can obtain the spectral resolution of $L$ by putting the spectral resolutions of the $L_k$'s into the formula (6.1).

7. Weyl law and equipartition of energy

7.1. Weyl law

Let us consider the spectral decomposition $(u_j, \omega_j)$ with $L u_j = \omega_j^2 u_j$, $j = 1, \cdots$, and $u_j$ an orthonormal basis of $L^2(X, \mathbb{R}^3)$.

**Theorem 7.1.** — (The Weyl law) If $N(\omega) := \# \{ j | \omega_j \leq \omega \}$ we have

$$N(\omega) \sim N_P(\omega) + N_S(\omega)$$

with

- $N_P(\omega) := \frac{1}{6\pi^2} \text{vol}(X) \left( \frac{\omega}{c_P} \right)^3$
- $N_S(\omega) := \frac{2}{6\pi^2} \text{vol}(X) \left( \frac{\omega}{c_S} \right)^3$

The previous theorem says that the asymptotic behaviour of the eigenvalues is the same as if we had 3 decoupled scalar operators, one of symbol $c_P^2 \| \xi \|^2$ and 2 of symbol $c_S^2 \| \xi \|^2$ acting on functions.

We have a more precise result (usually called the local Weyl law):

**Theorem 7.2.** — For any homogeneous symbol $a \in C^\infty(T^*X, \text{Sym}(\mathbb{R}^3))$ compactly supported in $x \in X$,

$$\sum_{\omega_j \leq \omega} \langle \text{Op}(a) u_j | u_j \rangle \sim (2\pi)^{-3} \left( \int_{c_P \| \xi \| \leq \omega} \text{trace}_P(a) |dx| + \int_{c_S \| \xi \| \leq \omega} \text{trace}_S(a) |dx| \right)$$

with $\text{trace}_P(a) = \text{trace}(\pi_P a)$ and similarly for $\text{trace}_S(a)$.

In particular
COROLLARY 7.3. — For any domain $D \subset \text{Int}(X)$, we have

$$\sum_{\omega_j \leq \omega} \int_D \|u_{j,P}\|^2 |dx| \sim \frac{1}{6\pi^2} |D| \left( \frac{\omega}{c_P} \right)^3,$$

$$\sum_{\omega_j \leq \omega} \int_D \|u_{j,S}\|^2 |dx| \sim \frac{2}{6\pi^2} |D| \left( \frac{\omega}{c_S} \right)^3,$$

$$\sum_{\omega_j \leq \omega} \int_D \langle u_{j,P} | u_{j,S} \rangle |dx| = O(\omega^2).$$

A very basic question is to understand how the dynamics produces a random wave in the previous setting. Let us give a

DEFINITION 7.4. — Let us take a solution of the wave equation $u(x,t)$ and a frequency $\Phi_\omega$ cut-off around the frequency $\omega$. We will say that $u(x,t)$ is equipartited if we have for large $\omega$:

$$\lim_{t \to \infty} \frac{\int_D \|\Phi_\omega u(x,t)\|^2 |dx|}{\int_X \|\Phi_\omega u(x,t)\|^2 |dx|} = \frac{|D|}{|X|},$$

$$\lim_{t \to \infty} \frac{\int_D \|\Phi_\omega u_S(x,t)\|^2 |dx|}{\int_D \|\Phi_\omega u_P(x,t)\|^2 |dx|} = \frac{2c_P^3}{c_S^3}.$$

Remark 7.5. — From the experimental point of view, only the second limit can be measured.

There are in fact 2 cases:

(1) The case $u = u_j(x) e^{i\omega_j t}$. Do the large frequencies individual eigen-modes satisfy an equipartition property?

(2) The case where the Cauchy datas of $u$ are localized at some point $x_0$. This case is much more interesting from the physical point of view. It corresponds to an earthquake with a source at $x_0$.

In the case of the (scalar) Laplace operator, both cases of the question have satisfactory answers:

- The first one is the celebrated Schnirelman Theorem (1974) which says that the answer is yes if the geodesic (billiard) flow is ergodic [12, 16, 3, 5, 17]
- The second one is a recent result by Roman Schubert [13] which says that the answer is yes if the geodesic (billiard) flow is Anosov and if $t$ is of the order of the so-called Heisenberg time. It uses the fact that Anosov systems are mixing.
7.2. A statistical interpretation of Weyl laws: the microcanonical ensemble

On any Hilbert space $H = (\mathcal{H}, \langle . | . \rangle)$, there is a canonical random field $w_H$, the white noise of $H$, which satisfies:

$$\forall e, f \in \mathcal{H}, \mathbb{E}(\langle w_H | e \rangle \langle w_H | f \rangle) = \langle f | e \rangle.$$ 

If $\dim H = \infty$, this field is not a vector of $\mathcal{H}$ but a “distribution”: if $A : H \rightarrow K$ is an Hilbert-Schmidt operator, $Aw_H$ is a random field on $K$.

Let us give some $E > 0$ and consider the Hilbert space $H_E \subset L^2(X, TX)$ which is generated by the eigenfields of frequency less than $E$. The associated random fields $w_E$ can be split as $w_E = w_{E,S} + w_{E,P}$.

8. The classical limit

Let us fix the frequency $\omega > 0$. Inside $X$, we have 2 decoupled dynamics: they are given by the Hamiltonians $h_P = c_P \|\xi\|$ (resp. $h_S = c_S \|\xi\|$) on the energy surfaces $\Sigma_P = \{h_P = \omega\}$ (resp. $\Sigma_S = \{h_S = \omega\}$). More precisely, we will introduce an extended phase space $Z := Z_S \cup Z_P$ with a measure $dm$ where

- $Z_S$ is the total space of the projective fiber bundle $P(E_S) \rightarrow \Sigma_S$ with the dynamics given by $\Phi_t^S(z, < v >) = (\phi_t^S(z), < T_t(v) >)$ with $\phi_t^S$ the Hamiltonian flow of $h_S$ and $T_t$ the parallel transport in the fiber bundle $E_S$ (the “Berry” phase). $Z_S$ is equipped with the measure $dm_S = dL_S \otimes dh$ where $dL_S$ is the microcanonical Liouville measure and $dh$ is the uniform measure on $P(E_S)$ with total mass 2.
- Similarly $Z_P = \Sigma_P$ and $dm_P$ is the microcanonical Liouville measure and the Hamiltonian dynamics $\varphi_t^P$ associated to $h_P$.

Of course the previous dynamics is not yet defined when hitting the boundary. At that point, we will define it as a Markov process whose probability transitions are given by the squares $\pi_P$, $\pi_S$ of the entries of the reflection matrix. Let us remark that the component of an $S-$wave tangent to the boundary (and normal to the ray) is reflected without adding a $P-$component. On the other hand the incoming and outgoing points $(y, \xi^\pm)$, $y \in \partial X$ are related by $\xi^+_|_{T_yX} = \xi^-|_{T_yX}$.

Let us check that the measure $dm$ is invariant by the previous stochastic process. Let us assume that $X_0(\omega)$ follows the probability $dm$:
\[ \mathbb{P}(\{X_0 \in U\}) = m(U) \quad \text{for any} \ U \subset Z. \] We need to check that, for all \( t > 0, \)
\[ \mathbb{P}(\{X_t \in U\}) = m(\{X_0 \in U\}). \]
Let us do that for small \( t \) and \( U \): let us consider for example \( U \subset Z_P \) and \( t \) so that the only way that \( X_t \in U \) is one reflection at some point \( y \in \partial X \). Either \( X_0 \in U_S \) or \( X_0 \in U_P \). In the first case \( X_0 = (z_0, \theta) \) and we can fix that \( \theta = 0 \) corresponds to a polarisation in the plane generated by the ray and the normal to the boundary. We get
\[ \mathbb{P}(\{X_t \in U\}) = \int_{U_P} \pi_P dm_P + \frac{2}{\pi} \int_{U_S} \pi_S \cos^2 \theta dL_S d\theta. \]
We get, using
\[ \frac{2}{\pi} \int_0^\pi \cos^2 \theta \, d\theta = 1, \]
\[ \mathbb{P}(\{X_t \in U\}) = \int_{U_P} \pi_P dm_P + \int_{U_S} \pi_S dm_S \]
and using the fact that the microlocal Liouville measures are conserved by the reflection process and that \( \pi_P + \pi_S = 1 \), we get the result. One can check a similar result if \( U \subset Z_S \).

In order to use the usual definitions of dynamical systems, we introduce the set \( T \) of trajectories \( z(t), \ t \in \mathbb{R} \) with the measure \( dP \) given as usual on cylindrical sets. The measure \( dP \) is invariant by the translation along trajectories. So we can say that the classical limit is ergodic or mixing.

### 8.1. Conjectures

- In the spirit of Schnirelman’s Theorem, we can ask the following:

  **Question 8.1.** — *If the classical random walk is ergodic, do we have equipartition of energies for a density one subsequence of eigenmodes \( u_{ji} \)?*

  Let us remark that the previous property is not valid if the manifold has no boundary. This conjecture is probably not true. A similar conjecture for another model with a probabilistic classical dynamics is not valid: the case of the Laplace operator on a generic star graph for which Schnirelman Theorem does not hold (see [7] for explicit formulae for eigenfunctions). At least the usual proof uses Egorov’s Theorem which is not true in this context!
• In the spirit of Schubert Theorem, I put the following

**Conjecture 8.2. — Let us assume that the classical limit is mixing and has some kind of hyperbolicity. If the Cauchy data of the Lamé wave equation are localized, there is a dynamical equipartition (large time):**

\[ u(x, t) \text{ satisfies equipartition for } t \text{ large but smaller than Ehrenfest times (to be defined)}. \]

9. **Appendix A: spectral resolutions, density of states and microcanonical ensembles**

Let \( L \) be a positive self-adjoint operator on \( L^2(X, \mathbb{R}^N) \). The spectral resolution \( de(x, y, \omega) \) of \( \sqrt{L} \) is a measure with values in \( \text{Sym}(\mathbb{R}^N) \) which satisfies:

\[ \delta(x - y) = \int_{\mathbb{R}} de(x, y, \omega) \]

and, for any function \( \Phi \) of \( L \):

\[ [\Phi(L)](x, y) = \int_{\mathbb{R}} \Phi(\omega^2) de(x, y, \omega). \]

The density of states is a measure on \( X \times \mathbb{R} \) defined by \( d\sigma(x, \omega) = \text{trace}(de(x, x, \omega)) \). In particular, we have, in case of a discrete spectrum:

\[ \#\{\omega_j \leq \omega\} = \int_X dx \int_0^\omega d\sigma(x, \omega'). \]

The **microcanonical ensemble** associated to an interval \( I = [\omega_-, \omega_+] \) is the white noise on the Hilbert space \( \Pi_I(L^2(X, \mathbb{R}^N)) \). We have then

\[ \mathbb{E}(\int_D |u(x)|^2 dx) = \int_D dx \int_{\omega \in I} d\sigma(x, \omega). \]

In other words, the expected energy of a mode whose frequency lies in \( [\omega_-, \omega_+] \) is given in terms of the density of states.

More generally, we have

\[ \mathbb{E}_I(\langle Au | u \rangle) = \text{Trace} A\Pi_I. \]

This can be evaluated in the large frequency regime if \( A \) is a \( \Psi DO \) as the trace of the pseudo-differential operator \( A\Pi_I \) of symbol \( a \circ \chi_I(\sqrt{L}) \).
BIBLIOGRAPHY


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